

Criterion for the critical number of fermions and chiral symmetry breaking in anisotropic QED₃

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By analyzing the strength of a photon-fermion coupling using basic scattering processes we calculate the effect of a velocity anisotropy on the critical number of fermions at which mass is dynamically generated in planar quantum electrodynamics. This gives a quantitative criterion which can be used to locate a quantum critical point at which fermions are gapped and confined out of the physical spectrum in a phase diagram of various condensed-matter systems. We also discuss the mechanism of relativity restoration within the symmetric quantum-critical phase of the theory.

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I. INTRODUCTION

Quantum electrodynamics (QED), the quantum theory of radiation and its interaction with matter, is a subject with far reaching impact in physics, from the calculation of the magnetic moment of the electron¹ to the widespread use of diagrammatic tools in every corner of theoretical physics. However, despite this long tradition as a central paradigm in physics, QED continues to confront us with many new challenges as its modern reincarnations emerge as effective theories of strongly correlated many-body problems, from quantum spin liquids to high-temperature superconductivity to graphene. Back in Feynman's day, we would have been very surprised if someone were trying to solve a relativistic problem with two speeds of light. These days, such problems are actually ubiquitous in modern theoretical physics, one example being the effective theory of low-energy excitations in a correlated d -wave superconductor. That theory is equivalent to an *anisotropic* quantum electrodynamics in 2+1 dimensions,^{2,3} in which different "speeds of light" appear naturally. Furthermore, there are other systems in which the low-energy description reduces to different versions QED₃, such as various forms of quantum spin liquids^{4,5} or the physics of graphene layers.⁶ All of these different problems share a common feature: they have a nodal structure that resembles a relativistic spectrum for the low-energy excitations. However, a distinctive feature of the QED₃ particularly relevant for cuprate superconductors is that it contains a significant intrinsic anisotropy, which exists due to the difference between the Fermi and gap velocities (v_F, v_Δ).

It is known that the *fermionic* anisotropy, $\alpha = v_F/v_\Delta$, is irrelevant in the perturbative renormalization-group (RG) sense,⁷ as long as the system is in the *symmetric* phase of QED₃. This is the quantum-critical phase of the theory, in which strongly interacting massless fermions acquire anomalous power-law behaviors in their various correlation functions. The anomalous dimension exponents characterizing this unusual state are universal and typically depend only on the total number of fermion flavors N . This however is true only as long as $N > N_c$, where N_c is the critical number of fermions at which the fermion mass is dynamically generated via the mechanism of spontaneous chiral symmetry breaking (CSB). Once CSB takes place, the fermions are gapped and confined out of the physical spectrum. This her-

alds a different massive phase of the theory which typically translates to a different state in the underlying condensed-matter system. The results of Ref. 7 are generally valid for arbitrary anisotropy as long as the number of fermions N is sufficiently large or for small anisotropy when N is greater than N_c of the isotropic case.

Evidently, N_c is an important number within the theory, not in the least because antiferromagnetic order in effective theories of high-temperature superconductors and quantum spin liquids generically arises via the above nonperturbative phenomenon of CSB—for example, in the context of cuprates, the chiral mass generation corresponds to the onset of a spin-density-wave order from within a quantum-disordered d -wave superconductor^{8,9} while it describes the formation of the Néel antiferromagnetic state and a whole family of other order states in the context of quantum spin liquids.^{4,5} Unlike the exponents of the critical massless phase, however, N_c itself is not universal. Consequently, an important question arises within the anisotropic QED₃: to what extent is the critical number of fermions, N_c , which defines the boundary between broken and unbroken chiral symmetry, affected when such anisotropy is present?

In this paper, our goal is to provide an answer to this question. Of course, ours is not *the* answer, for two reasons. First, since QED₃ is a strongly interacting theory its exact behavior is beyond our reach. Second, since N_c is not universal there are actually *many different* N_c 's: the nominally irrelevant couplings within the QED₃ theory of a quantum-disordered d -wave superconductor^{8,9} are very different than those of lattice-based quantum spin liquids.^{4,5} Furthermore, both these N_c 's are different from the *intrinsic* N_c of the pure QED₃ field theory considered here (defined through Balaban-Jaffe regularization,¹⁰ for example). However, all these issues notwithstanding, even in the absence of the exact solution, it is still possible to make a rather accurate determination of the *parametric* dependence of N_c on the anisotropy, once an "exact" N_c is known for the isotropic case from a different source, say, from numerical simulations. Our goal is to devise a criterion for determination of N_c within the anisotropic QED₃ which, while not exact, still provides a rather accurate description of how the CSB boundary changes as a function of the parameters of the theory. The philosophy here is similar to the one behind the ubiquitous Lindemann¹¹ criterion, originally proposed to predict the melting point of a solid. While not exact, the Linde-

mann criterion has proven itself a remarkably accurate and useful in a wide range of situations, from classical to quantum solids, from Wigner crystals to Abrikosov vortex lattices.

To devise our criterion, we point out that mass generation is a consequence of the fermion-photon coupling strength in QED. This strength is generally renormalized from its bare value by virtual polarizability of the vacuum. Relying on this fact, we propose a natural way to measure the strength of the gauge field by focusing on the matrix element that represents processes in which one photon is exchanged between two fermions. We stipulate that the CSB and mass generation take place when this matrix element exceeds certain critical value. Within the isotropic QED₃ this is manifestly an exact statement—the only unknown is the actual value of N_c which we can either infer from a separate calculation or borrow from numerical simulations.^{12–14} Once our “Lindemann criterion” is calibrated in this fashion, we proceed to evaluate the appropriate matrix element in the anisotropic theory and propose that the CSB takes place when this matrix element exceeds the *same* critical value.

Following the above procedure, our criterion allows us to derive the following main results: first, we show that N_c is not just a function of the anisotropy α but also a nonmonotonic function of v_F and v_Δ themselves. The reason behind this peculiar behavior is the existence of a third relevant velocity in the theory, the speed of light, c_s , which naturally appears in the Maxwellian action for the gauge field. We confirm our findings for $v_\Delta > c_s$ by using the Schwinger-Dyson (SD) equation within the Pisarski approximation;¹⁵ both results show the same functional dependence on v_Δ in this regime. Another result is that the critical number of fermions decreases even for $\alpha=1$, as long as $v_F=v_\Delta \gg c_s$. Thus, in the context of the QED₃ effective theory of underdoped cuprates, we generically expect a nonsuperconducting, nonmagnetic pseudogap state to intrude prior to the emergence of antiferromagnetism as d -wave superconductor is underdoped toward half filling.

II. CRITERION AND CALCULATION

We start by defining the form of the anisotropic QED₃ Lagrangian and the main physical quantities that will be used in our calculations. The Lagrangian is

$$\mathcal{L}_{\text{Aniso}} = \sum_{l=1}^N \bar{\psi}_l v_\mu^{(l)} \gamma_\mu (i\partial_\mu - a_\mu) \psi_l + \mathcal{L}[a_\mu(x)], \quad (1)$$

where $\bar{\psi}_l$ are four component spinors and a_μ is a gauge field. In condensed-matter environment, Eq. (1) is the low-energy effective theory of some strongly correlated electron system. For example, in a phase-disordered d -wave superconductor, the gauge field a_μ couples nodal Bogoliubov–de Gennes–Dirac fermions to fluctuating quantum vortex-antivortex pairs through a Maxwellian action $\mathcal{L}[a_\mu(x)]$. γ_μ are the standard gamma matrices, such that $\{\gamma_\nu, \gamma_\mu\} = \delta_{\mu\nu}$ and $v_\mu^{(l)}$ is defined as $(1, v_F, v_\Delta)$ and $(1, v_\Delta, v_F)$ for nodes $l=1, 2$, respectively. Fermi and “gap” velocities, v_F and v_Δ , define fermionic anisotropy at each node.

To make further progress we first focus our attention on a particularly simple case: the noninteracting one ($a_\mu=0$). This case is equivalent to saying that the phase is rigid or that the departure point in our analysis is the fully ordered superconducting state. In that case Eq. (1) is reduced to

$$\mathcal{L}_{\text{Dirac}} = \sum_{l=1}^2 \bar{\psi}_l v_\mu^{(l)} \gamma^\mu (i\partial_\mu + m) \psi_l, \quad (2)$$

where a mass term has been introduced only for normalization purposes. In the end our results will be robust in the $m \rightarrow 0$ limit.

The equations of motion for these noninteracting Dirac fermions are

$$(-v_\mu^{(l)} \gamma^\mu i\partial_\mu + m) \psi_l = 0 \quad (3)$$

with $l=1, 2$. From this expression it is clear that the task of solving Eq. (3) can be performed independently for each fermion flavor 1 and 2. Now, if we focus on $l=1$, we can reduce the anisotropic equation of motion to the isotropic one using a simple change in variables $\tau' = \tau$, $x' = v_F x$, and $y' = v_\Delta y$. It is worth remarking that the intrinsic fermionic anisotropy *cannot* be removed in the case of the real d -wave superconductor once the gauge field is present, as can be seen from the definition of $v_\mu^{(l)}$.

We now outline how a typical perturbative approach for computing the effect of coupling to the gauge field works: starting from the solutions obtained for the problem in which $a_\mu=0$, namely, $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ —both having the anisotropy encoded in their spatial oscillations—we will perturbatively introduce the field a_μ . First, we choose our free solutions for each node in the form $\psi_l^+(\vec{x}) = s_r(m, 0) \exp(-imt)$ and $\psi_l^-(\vec{x}) = t_r(m, 0) \exp(+imt)$, where for convenience we have defined $s_1(m, 0) = (1, 0, 0, 0)$, $s_2(m, 0) = (0, 1, 0, 0)$, $t_1(m, 0) = (0, 0, 1, 0)$, and $t_2(m, 0) = (0, 0, 0, 1)$. The eigenstates of Eq. (3) are then

$$\psi_l(x) = \sum_{\vec{p}r=1,2} \left(\frac{m}{VE_{\vec{p}}} \right)^{1/2} [b_{r\vec{p}} \bar{s}_r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + d_{r\vec{p}} \bar{t}_r(\vec{p}) e^{i\vec{p}\cdot\vec{x}}], \quad (4)$$

where $b_{r\vec{p}}$ and $d_{r\vec{p}}$ are standard fermionic annihilation operators, while \vec{p} and r are the momentum and spinor indices, respectively. In the case of decoupled nodes, when $a_\mu=0$, all the information about the anisotropy is encoded in the exponential factor. Now, for finite momentum we can use the ansatz $\psi_1^+(\vec{x}) = s_r(\vec{k}) \exp(-i\vec{k}\cdot\vec{x}')$ and $\psi_1^-(\vec{x}) = t_r(\vec{k}) \exp(+i\vec{k}\cdot\vec{x}')$, where $\vec{x}' = (\tau, v_F x, v_\Delta y)$ for node 1. The spinors $s_r(\vec{k})$ and $t_r(\vec{k})$ can be found directly from the equation of motion

$$(\vec{k} - m) s_r(\vec{k}) = 0, \quad (\vec{k} + m) t_r(\vec{k}) = 0, \quad (5)$$

where $\vec{k} = \gamma^\mu k_\mu$. Thus, applying $(\vec{k} + m)$ and $(\vec{k} - m)$ to $s_r(m, 0)$ and $t_r(m, 0)$, respectively, we can generate the needed solutions.

Using these free fermions in the Heisenberg picture, we now introduce the gauge field by employing the well-known formal solution of the scattering matrix S , equivalent to

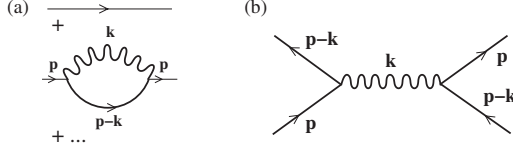


FIG. 1. (a) Diagrammatic form of the Schwinger-Dyson equation at one-loop approximation. (b) Diagram used to measure the strength of a photon.

$$S = T \left\{ \exp \left[-i \int d^{2+1}x \mathcal{H}_I(\vec{x}) \right] \right\}, \quad (6)$$

where T and $:$ are the time-ordered operator and the normal order operation, respectively. The interaction Hamiltonian density that couples fermion fields to the gauge field is given by

$$\mathcal{H}_I(\vec{x}) = \sum_{l=1}^2 : \bar{\psi}_l v_\mu^{(l)} \gamma_\mu a_\mu \psi_l :. \quad (7)$$

Now, as usual, we want to analyze the matrix element $\langle f|S|i\rangle$, where both the final and initial state are assumed to be eigenstates of the *unperturbed* Hamiltonian. Using the Heisenberg representation of the free fields $\psi(\vec{x})$ and the Wick's theorem, we recognize that the only relevant diagrams are the interactions photon-fermion and fermion-fermion, the so-called direct scattering and the Möller scattering. The calculation simplifies here because, due to its fluctuating nature, the contribution of a_μ to the first term of the S matrix expansion forces the said matrix element to vanish. Consequently, the leading contribution relevant for our purposes comes from the second-order term where the contraction of $a_\mu(\vec{x}_1)$ with $a_\nu(\vec{x}_2)$ is nothing else but the photon propagator $D_{\mu\nu}(\vec{x}_1 - \vec{x}_2)$, which is the real-space *inverse* of the polarization function

$$\Pi_{\mu\nu} = \sum_n \frac{N}{16v_F v_\Delta} \sqrt{k_\alpha g_{\alpha\beta}^{(n)} k_\beta} \left(g_{\mu\nu}^{(n)} - \frac{g_{\mu\rho}^{(n)} k_\rho g_{\nu\lambda}^{(n)} k_\lambda}{k_\alpha g_{\alpha\beta}^{(n)} k_\beta} \right), \quad (8)$$

where the diagonal nodal “metric” $g_{\mu\nu}^{(n)}$ is defined⁷ by $g_{00}^{(1)} = g_{00}^{(2)} = 1$, $g_{11}^{(1)} = g_{22}^{(2)} = v_F^2$, and $g_{22}^{(1)} = g_{11}^{(2)} = v_\Delta^2$, and $\gamma_\nu^{(n)} = \sqrt{g_{\mu\nu}^{(n)}} \gamma_\mu$. We will now demonstrate that this second-order element of the perturbation theory can be used to gainfully define the “strength” of the fermion-photon interaction and determine the critical number of fermionic flavors $N_c(\alpha)$, without ever solving the Schwinger-Dyson equation, even though a perturbative approach itself is condemned to failure.

In order to accomplish this, let us focus on the mass-generation problem. It is well known that Schwinger-Dyson equation, Fig. 1(a), has a nontrivial solution ($m > 0$) once the initially soft photon (factor $1/N$ in the large N limit) becomes harder at a critical number of fermions N_c with $3 < N_c < 5$.^{12,15–17} That means for $N < N_c$ mass will be dynamically generated and therefore chiral symmetry would be broken.

The existence of anisotropy makes analyzing the birth of a gapped state through the mechanism of mass generation using the Schwinger-Dyson formalism almost impossible

with the noteworthy exception of the isotropic limit and the so-called small anisotropy case ($v_F = 1 + \delta$ and $v_\Delta = 1$, where $\delta \ll 1$).^{7,18,19} Our aim is to explore mass generation for arbitrary anisotropy because this is the physically relevant regime. Our technique is based on the observation that as we lower N we are effectively making the photon interaction stronger. This is clear because the photon propagator contains a screening factor $\frac{1}{N}$. From this simple observation we can see that mass generation is a phenomenon that is intrinsically tied to the strength of the gauge field (photons). Put simply, we can translate the meaning of N_c into the following statement, which clearly is a necessary condition for mass generation: if the photon-exchange interaction is stronger than some threshold value A_c then the fermion mass will be dynamically generated. This assertion clearly lacks the precision of a mathematical theorem but we now proceed to demonstrate that it has the practical virtue of a useful criterion.

The problem that we are facing is straightforward: how can we usefully quantify the *strength* of the gauge field? The simplest way to measure the strength of a photon is to analyze a process in which one photon is exchanged. For the sake of simplicity, we will analyze the *fermion-fermion* scattering mediated by one *photon*, as shown in Fig. 1(b) which corresponds to the second-order term in perturbation theory. Thus using the N_c borrowed from the isotropic limit, we define $A_c = \langle i|S^{(2)}|f\rangle[N_c(\alpha=1)]$, the matrix element of the S matrix evaluated at the isotropic limit, as the natural candidate to measure the strength of the photon. We can safely say that in an anisotropic theory mass will be generated at the critical strength defined by A_c .

Thus solving the equation, $A_c = \langle i|S^{(2)}|f\rangle[N_c(\alpha)]$ for $N_c(\alpha)$, we will find the influence of anisotropy on the critical number of fermion flavors. In the isotropic case, the choice of node will not make any difference, however in the anisotropic case the situation will be quite different. Regarding the symmetries of the problem we must analyze the invariant, $\langle i|S^{(2)}|f\rangle(v_F, v_\Delta) + \langle i|S^{(2)}|f\rangle(v_\Delta, v_F)$ because nodal particles can be born indifferently in nodes 1 or 2. This quantity is invariant under the transformation $v_F \leftrightarrow v_\Delta$, which is a physical symmetry of this problem.

We now proceed to compute the first nonzero matrix element $\langle i|S^{(2)}|f\rangle$. The second-order process following from the S -matrix term, Eq. (6), is

$$S^{(2)} \propto T \left[\int \mathcal{H}_I(\vec{x}_1) \mathcal{H}_I(\vec{x}_2) dx_1 dx_2 \right], \quad (9)$$

which can be written (integration with respect to x_1 and x_2 is implicit) as a sum of terms of the type

$$T[: \bar{\psi}_1(x_1) \gamma_\mu^{(1)} a_\mu(x_1) \psi_1(x_1) :: \bar{\psi}_1(x_2) \gamma_\nu^{(1)} a_\nu(x_2) \psi_1(x_2) :]. \quad (10)$$

By applying Wick's theorem to the previous expression terms with and without contractions are obtained. However, due to the fluctuating nature of the gauge field most of those terms will vanish. The survivor will be the one containing the contraction of the gauge field with itself. Thus, the problem is reduced to compute the matrix element

$$: \bar{\psi}_1(x_1) \gamma_\mu^{(1)} \psi_1(x_1) \bar{\psi}_1(x_2) \gamma_\nu^{(1)} \psi_1(x_2) D_{\mu\nu}(x_2 - x_1) :. \quad (11)$$

To evaluate such matrix element we must define the initial and final states of the system $|i\rangle$ and $|f\rangle$. As the strength of the photon should be independent of the states that we use to measure it, we are free to use

$$|i\rangle = b_{r_1 \vec{p}_1}^\dagger b_{r_2 \vec{p}_2}^\dagger |0\rangle \quad \text{and} \quad |f\rangle = b_{r'_1 \vec{p}'_1}^\dagger b_{r'_2 \vec{p}'_2}^\dagger |0\rangle, \quad (12)$$

which we already know from the free theory. The relevant matrix element is

$$\begin{aligned} \frac{1}{N} \langle i | S^{(2)} | f \rangle (\Lambda, v_F, v_\Delta) &\sim \frac{1}{v_\Delta v_F} \int d^3 k \left(\frac{1}{p_{i0}(p_{i0} - k_0)} \right) \\ &\times Q_\mu(\vec{p}_i - \vec{k}, \vec{p}_i) D_{\mu\nu}(g^{(1)} \vec{k}) \\ &\times Q_\nu(\vec{p}_i, \vec{p}_i - \vec{k}), \end{aligned} \quad (13)$$

where $Q_\mu(\vec{P}, \vec{p}) = \bar{s}(\vec{P}) \gamma_\mu^{(1)} s(\vec{p})$ with $\bar{s} = s \gamma_0$. The integral is to be performed over all the allowed k space but we have one constraint that is the spectrum of the Dirac fermions, $k_0 = \sqrt{(v_F k_x)^2 + (v_\Delta k_y)^2}$, which allows us to further reduce the evaluation of this integral.

The integration limits are fixed from the condition that the momentum transferred in the processes shown in Fig. 1(b) cannot be larger than the momentum of the incident particle, \vec{p}_i . We have set $p_x = p_y = \Lambda$ as an upper cutoff. We have also verified that the only effect of choosing a different cutoff for different spatial directions is that it makes the convergence slower. Thus, using the invariant measure of the strength of the gauge field, it is straightforward to show that the ratio between critical number of fermions at velocities v_F and v_Δ with respect to the isotropic point is

$$\beta = \frac{\langle i | S^{(2)} | f \rangle (\Lambda, v_F, v_\Delta) + \langle i | S^{(2)} | f \rangle (\Lambda, v_\Delta, v_F)}{2 \langle i | S^{(2)} | f \rangle (\Lambda, 1, 1)}. \quad (14)$$

Thus, the critical number of fermions for the anisotropic theory is given by the product of $\beta(v_F, v_\Delta)$ and the critical number of fermions at the isotropic point $N_c(1, 1)$. In principle, this measure of the strength of the gauge field depends on the initial momentum of the particles but it turns out that the ratio between the strength of the gauge field with anisotropy and the strength of the gauge field without anisotropy $\alpha = 1$ is not sensitive to the momentum of the incident fermions, \vec{p}_i . In fact, all the calculated quantities converge to a fixed value for a fairly low-momentum cutoff, Λ . It is important to remark that β is gauge invariant due to the transverse nature of the photon.

Indeed we can do better. By isolating the leading divergent behavior of the scattering amplitude we have obtained that in the $\Lambda \rightarrow \infty$ at the leading order in the large N approximation, the ratio β is given by

$$\begin{aligned} \beta(v_F, v_\Delta) &= \zeta_1 \left[\sqrt{1 + v_\Delta^2} \mathbf{E} \left(\frac{v_\Delta^2 - v_F^2}{1 + v_\Delta^2} \right) + \sqrt{1 + v_F^2} \mathbf{E} \left(\frac{v_F^2 - v_\Delta^2}{1 + v_F^2} \right) \right] \\ &+ \zeta_2 \left[\mathbf{K} \left(\frac{v_F^2 - v_\Delta^2}{1 + v_F^2} \right) + \mathbf{K} \left(\frac{v_\Delta^2 - v_F^2}{1 + v_\Delta^2} \right) \right], \end{aligned} \quad (15)$$

where

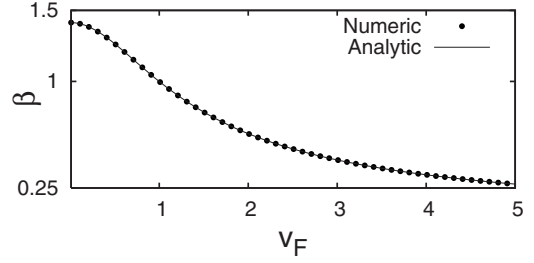


FIG. 2. Comparison between the numerical and analytical calculation of β in the case for which $v_F = v_\Delta$.

$$\mathbf{E}(m) = \int_0^{\pi/2} [1 - m \sin(\theta)]^{1/2} d\theta,$$

$$\mathbf{K}(m) = \int_0^{\pi/2} [1 - m \sin(\theta)]^{-1/2} d\theta,$$

$$\zeta_1 = -2 \left(\frac{1 - 2(v_F^2 + v_\Delta^2) - v_F^2 v_\Delta^2 + v_F^4 + v_\Delta^4}{3\sqrt{2}\pi v_F v_\Delta^2} \right),$$

$$\zeta_2 = \frac{2 - 3(v_F^2 + v_\Delta^2) + (v_F^2 - v_\Delta^2)^2 + v_F^4 v_\Delta^2 + v_\Delta^4 v_F^2}{3\sqrt{2}\pi v_F^2 v_\Delta^2 \sqrt{(1 + v_F^2)(1 + v_\Delta^2)}} \quad (16)$$

are the elliptic functions of first and second kind, respectively, and the prefactors are polynomials that are symmetric under $v_F \leftrightarrow v_\Delta$. Thus β is explicitly invariant under the transformation $v_F \leftrightarrow v_\Delta$ as it must be.

An important case is the one in which there is no *fermionic anisotropy*, i.e., $v_F = v_\Delta$. In this case all the integrals can be performed and the full divergent behavior of Eq. (13) can be isolated by going to cylindrical coordinates as the $x \leftrightarrow y$ symmetry suggests. The result is just the limit behavior of Eq. (15) when taking $v_F = v_\Delta$. In that limit the dependence on the fermionic velocities becomes particularly simple,

$$\beta = \sqrt{\frac{2}{1 + v_\Delta^2}}. \quad (17)$$

To test the accuracy of our technique, the result is compared with the exact numerical integration in Fig. 2, showing excellent agreement. For low speeds $v_\Delta < 1$, our result suggests that the critical number of fermions increases. We discuss this case in more detail below. On the opposite limit when the gauge-field velocity is small compared with the other two velocities the critical number of fermions goes to zero.

A nontrivial dependence on the gauge-field velocity was also found; there are three different regimes depending upon the value of v_F , with respect to c_s , as shown in Fig. 3. For $v_F < c_s$, the critical number of fermions increases, reaching unexpected values for large anisotropy, leading to a gapped state in this sector. On the other hand for $v_F > c_s$, N_c decreases and we can generically expect gapless excitations. The most unexpected result occurs if $v_F = c_s$; in this case $N_c(\alpha) \approx N_c(1)$ for $\alpha > 1$. Even though we found small deviations with respect to the isotropic value near to $\alpha = 2$, they do not change the integer part of N_c (see also Fig. 4). Thus, in

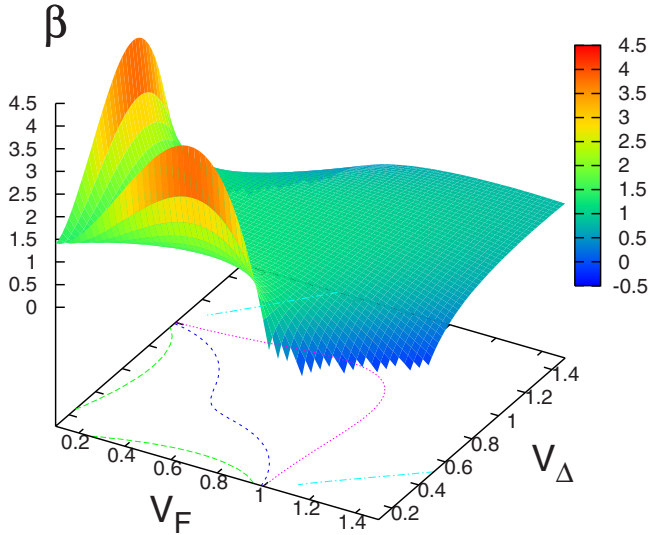


FIG. 3. (Color online) The calculated ratio β . The minimum value of v_F and v_Δ is 0.15 and the maximum is 1.5. This graph shows that N_c is not a simple function of α but a more complex function of v_F and v_Δ .

this regime large anisotropy will be completely irrelevant.

Using $N_c(1)=4$, which is the integer closest to the gauge-invariant critical number of fermions $128/3\pi^2$ found by Nash,¹⁶ we have obtained a phase diagram in the (v_F, v_Δ) space, Fig. 5. In this plot it is clear that if both velocities are smaller than c_s , the critical number of fermions never goes to zero. However, outside that square region $N_c \rightarrow 0$ for high anisotropies.

To show how important it is to consider the scale defined by c_s we have also plotted $\beta(\alpha)$ for different values of v_F (0.5, 1, and 5.0) that we kept fixed as v_Δ was varied (see Fig. 4). In this plot we make evident that c_s is a relevant parameter which we are allowed to set to 1, however the detailed behavior of the system is not a simple function of the anisotropy α but an explicit function of both parameters v_F and v_Δ .

In order to test our results we compare them with recent numerical work that has been done by Thomas and Hands (T-H) (Ref. 12) in an attempt to understand how anisotropy can modify the properties of QED₃. The lattice (Euclidean) version used by T-H includes an anisotropy which is in-

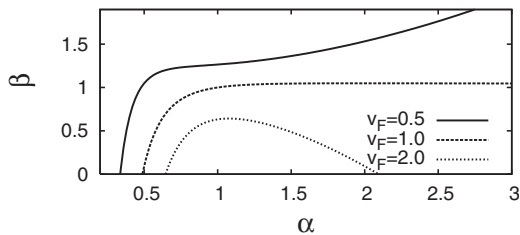


FIG. 4. Ratio between the critical number of fermions in the anisotropic theory and its equivalent within the isotropic theory as a function of anisotropy, $\beta(\alpha)$. The three curves represent the three regimes found: $N_c(\alpha) < N_c(1)$ if $v_F > 1$, $N_c(\alpha)/N_c(1)$ approaches a constant ~ 1 if $v_F=1$, and $N_c(\alpha)$ increases for increasing α if $v_F < 1$.

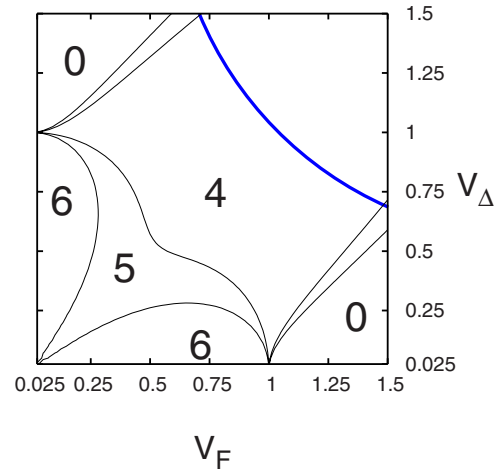


FIG. 5. (Color online) Phase diagram for the critical number of fermions in the velocity space. We have set the critical number of fermions for the isotropic theory at $N_c=4$. Inside the square defined by the lines $v_F=1$ and $v_\Delta=1$ the critical number of fermions never vanishes. The thick blue line corresponds to $v_F=1/v_\Delta$ and represents the one-dimensional domain simulated by Thomas and Hands (Ref. 12).

tended to mimic the continuum model in the incarnation presented by Lee and Herbut.¹⁸ The QED₃ theory presented in Lee's article is parametrized in terms of two quantities $\delta = \sqrt{v_F v_\Delta}$ and $\lambda = v_F/v_\Delta$. T-H have used an extended lattice model similar to the one used by Dagotto *et al.*²⁰ that in the continuum limit resembles the behavior of Eq. (1). In order to perform the simulation the action used was

$$S = \sum_{i=1}^N \sum_{x,x'} a^3 \bar{\chi}_i(x) M_{x,x'} \chi_i(x') + \frac{\beta}{2} a^3 \Theta_{\mu\nu}^2(x), \quad (18)$$

where the anisotropy was introduced in the fermion matrix

$$M_{x,x'} = \frac{1}{2a} \sum_{\mu=1}^3 \xi_\mu(x) [\delta_{x',x+\hat{\mu}} U_{x\mu} - \delta_{x',x-\hat{\mu}} U_{x'\mu}^\dagger] + m \delta_{\mu\nu} \quad (19)$$

and

$$\xi_\mu(x) = \lambda_\mu \eta_\mu(x), \quad (20)$$

$$\eta_\mu(x) = (-1)^{x_1 + \dots + x_\mu}, \quad (21)$$

with $x_1=x$, $x_2=y$, and $x_3=\tau$ is the Kawamoto-Smit phase of the staggered fermion field. The lattice spacing is a . An important definition is that of the anisotropy factors, $\lambda_x = \alpha^{-1/2}$, $\lambda_y = \alpha^{1/2}$, and $\lambda_\tau = 1$. This definition is important for our purposes because it shows that T-H lattice theory does not keep the flavor symmetry of the model relevant for cuprates.^{2,3,7,12} Regardless of this intrinsic drawback of the method we will show that it still mimics the qualitative behavior of the flavor-symmetric anisotropic QED₃ at least in the low-anisotropy limit.

In their numerical simulation T-H have a single parameter, which is equivalent to $\alpha = v_F/v_\Delta$ and as an extra constraint they have set $\delta=1$. This choice implies that $v_F=1/v_\Delta$ and

therefore $\alpha=1/v_\Delta^2$. That means that T-H have simulated a one-dimensional domain of the whole parameter space (v_F, v_Δ) . That domain is shown in Fig. 5 as a thick blue line. In order to make a link between the amount of condensate $\langle \bar{\psi}\psi \rangle$ and its relation with $N_c(\alpha)$ we will assume that the functional form of the dynamically generated mass does not change^{16,17} as we introduce anisotropy in the system. This can be explicitly checked in the small anisotropy limit.¹⁹ As long as we are inside the broken phase and near to the boundary between broken and unbroken phases, the mass has the following functional form:

$$m = m_o \exp\left(\frac{-2\pi}{\sqrt{\frac{N_c(v_F, v_\Delta)}{N} - 1}}\right), \quad (22)$$

where m_o may depend on N . However, what is important is how to define the boundary between the massless phase and the massive phase. This is done by solving Eq. (22) when $m=0$. We can also interpret this equation in the following way: for a fixed number of fermions N this equation allow us to understand how the mass changes as a function of $N_c(\alpha)$ when N is close to N_c . In fact, for any number of fermions when $N_c(\alpha)=N$, it is prohibited to have any condensate. That means that the critical anisotropy α_c that solves $m(\alpha_c)=0$ for any value of N is the same as the critical anisotropy that solves $N_c(\alpha_c)=N$. Hands reported that such a critical anisotropy $\alpha_c \approx 4$ in a 16^3 -site lattice simulation, which is in agreement with the critical value found by us, $\alpha_c \approx 3$. This decreasing behavior is in contrast to the one found in Ref. 18, where it was claimed that N_c increases as a function of the bare anisotropy. Taking into account that the lattice simulation is a nonperturbative method that does not rely in any educated ansatz, T-H results strongly support our view of the phenomenon. Still, we should mention that the critical anisotropy calculated by T-H using the anisotropic scaling is not, in the strict sense, quantitatively accurate in the context of cuprates for two reasons. First, the scaling used breaks the crystal isotropy, or, in different words, their simulation is not invariant under flavor exchange. Second, they assumed that the two fermionic velocities change little around the gauge-field velocity $c_s=1$. Due to the collective nature of phase defects we expect that $c_s \ll v_F$ in cuprates.²¹

On the other hand lattice simulations are always performed in finite lattices and therefore the correct comparison with our results should be done by considering both an upper cutoff and a lower cutoff. In principle we are free to make the upper cutoff as large as we want but the lower cutoff dependence may be important when comparing our results with finite lattice simulations.²² Setting the upper cutoff as $\Lambda_u=1$ then the lower cutoff should be $\Lambda_d \sim 1/L$, where L is the size of the system. As we change the lower cutoff we have found no significant differences in our results as it goes to zero.

We have also investigated the effect of the breaking of the flavor symmetry in our scheme. For this purpose we have used only one amplitude for the photon so that the final kernel is not invariant under $v_F \leftrightarrow v_\Delta$. In this case, we have obtained the same qualitative behavior as shown in Fig. 6 and the small negative slope at the isotropic point, detail that is in agreement with the results shown by T-H in Fig. 5 (Ref.

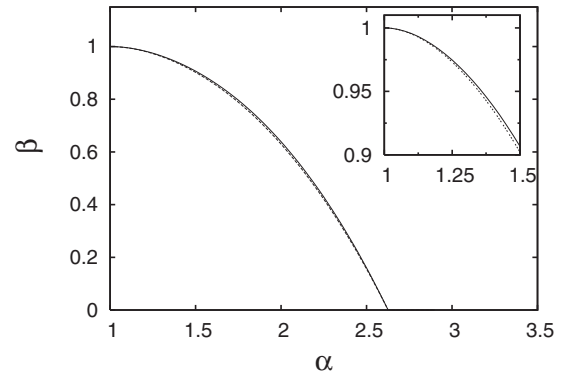


FIG. 6. β within the domain defined by the blue line of Fig. 5. The continuous line represents the flavor-symmetric calculation. The dotted line is the nonsymmetric calculation. For the latter case we have only plotted the real part because the breaking of the flavor symmetry introduces a small but persistent imaginary part.

11). This clearly is a symptom of the breaking of flavor symmetry. However, there is a very important issue that emerges once we arbitrarily break flavor symmetry. The function β starts to pick up a phase which is unphysical. β must be a real number as N_c is. This makes evident that even with the simplest interaction between fermions the flavor symmetry is needed in order to obtain a meaningful value of β . This observation suggests that numerical simulations that preserve flavor symmetry are the only reliable way to extract accurate critical values. However, we must enfore that the qualitative behavior of T-H simulation agrees with the physical picture proposed in this paper. Looking at the obtained phase diagram, Fig. 5, the question that naturally arises is, which regime is the physically relevant one? Clearly our answer will depend on the ratio v_F/c_s . At finite but still small temperature T we can use the continuum vortex-antivortex Coulomb plasma model in order to estimate the gauge-field velocity. Identifying the speed of light from the Maxwellian form of the action for the gauge field we find that $c_s \sim \sqrt{n_l}/T$ at finite T , where n_l is the density of vortices.³ Thus for any $T>0$ as we approach the superconducting state $n_l \rightarrow 0$ and thus $c_s \rightarrow 0$, resulting in a protected symmetric phase. As n_l increases, for small but finite T , c_s may reach very high values, but if those values are larger then v_F is unclear. If we go to the line $T=0$ quantum fluctuations will drive the system into a region in which the value of c_s will depend on the specific value of the dynamical critical exponent z , which in some simplified calculations was adopted to be 1. The more striking problem about identifying the precise value of c_s is that this velocity is a function of the correlation length $c_s = f(\xi_{sc})$ but at the same time we know that $\xi_{sc} = \xi_{sc}(x, T)$, thus a self-consistent treatment or knowledge of the correlation length from experiments will be needed to settle this problem and give an accurate phase diagram that identifies the relevance of these different regimes.

To provide further evidence of our findings we have also reanalyzed from a different perspective the case in which *fermionic anisotropy* is one. We have applied Pisarski's technique¹⁵ to find the qualitative behavior of the dynamically generated mass as a function of v_Δ .

We will assume that $\Sigma(p) \sim \Sigma(0)$. Thus in the appropriate integration interval $[\Sigma(0), \Lambda]$ the mass will be a constant.

This assumption is certainly incorrect, as was shown by Appelquist *et al.*^{16,17} However, it will allow us to compare the qualitative behavior of the mass as a function of v_Δ for the case in which $v_F=v_\Delta$. In this case the sums can be performed analytically by going to cylindrical coordinates instead of spherical ones. The integration was performed over the shell defined by a lower cutoff $\Sigma(0)$ and an upper cutoff Λ . Thus the Schwinger-Dyson equation

$$\Sigma_{(n)}(\vec{p}) = \int \frac{\gamma_\mu^{(n)} D_{\mu\nu}(\vec{p}-\vec{k}) \gamma_\nu^{(n)} \Sigma_{(n)}(\vec{k})}{k_\mu \delta_{\mu\nu} k_\nu + \Sigma_{(n)}^2(\vec{k})} d^3k \quad (23)$$

can be solved in this rough approximation and the result is

$$\Sigma(0) \sim \exp\left(-\frac{N\pi^2 v_\Delta^2}{8\sqrt{v_\Delta^2-1}}\right), \quad (24)$$

which is a real number as long as $v_\Delta > 1$. Thus as long as the *neutrinos* move faster than the photons the generated mass indeed does depend on v_Δ .

We expect that—given the fact that the decay factor in Pisarski's result is of order N_c —the correction found for that factor will give us the functional behavior of $N_c(v_\Delta)$. Thus, as $v_\Delta \gg 1$ we would expect that $N_c \sim 1/v_\Delta$. This is consistent with the result obtained from the proposed criterion, Eq. (17). However, we should warn the reader that even though a mathematical expression can be obtained for $v_\Delta < 1$ the nature of the system will change in this case, casting doubt on the validity of our criterion in that region. Indeed, from SD equation at the Pisarski-level approximation for low fermionic velocities, the self-energy will acquire an imaginary part which can be interpreted as leading to a confinement²³ for fermions of the theory. This instantly calls into question the validity of using a pure plane-wave-type solution for the computation of the scattering process.

In this approximation, the nature of the solution changes considerably at the point where $v_\Delta=1$. That is so because the radial integral gets a logarithmic contribution that is proportional to $\sqrt{1-v_\Delta^2}$, which overwhelms the leading contribution at the isotropic point. Alternatively, in more physical terms, if photons and massless fermions move at precisely the same speed this is “infinitely” different than having the photons that move faster than fermions. In the latter case it is natural to expect an “overscreening” behavior, in which constant exchange of fast photons ultimately leads to confinement. On the opposite side, with fermions moving faster than photons, we expect photons to be less effective in screening the fermions and thus less effective in generating their mass. We have also checked that in the isotropic limit the Pisarski's answer obtains and thus our results are not an artifact of the parametrization used.

So far, we have shown that mass generation has a nonuniversal behavior, which arises due to the breaking of Lorentz invariance. Thus, it is natural for a cautious reader to wonder if, once $N > N_c$, the renormalized effective low-energy theory is in fact Lorentz invariant or not. To begin with we emphasize that the fact that α flows to one by itself does not guarantee that the full Lorentz invariance will emerge unless both c_s and v_F are set equal; this in effect acts as an extra con-

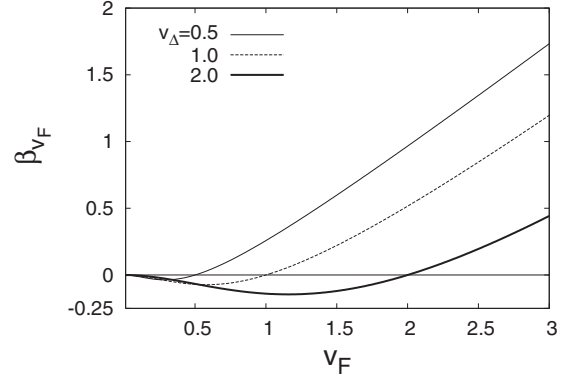


FIG. 7. The RG β function for v_F (in arbitrary units) computed using the same method as in Ref. 7 but without setting $v_\Delta=c_s=1$. This plot shows the parametric dependence of the fixed point on the value at which v_Δ is initially set in this calculation.

straint. This is shown in Figs. 7 and 8 where it is easy to see that even though $\alpha \rightarrow 1$ neither v_F nor v_Δ converge to $c_s=1$ unless the above-mentioned extra constraint is imposed. To show that the full Lorentz invariance is indeed restored we must prove that both fermionic velocities v_F and v_Δ flow to $c_s=1$ independently.

It is easy to see that Lorentz invariance can still be broken even with the fermionic anisotropy set to unity. Let us assume that the bare values of the Fermi and gap velocity are equal to each other but different from the gauge-field velocity, i.e., $v_F=v_\Delta$, but $v_F \neq c_s$. The Lagrangian for this simplified theory is^{24,25}

$$\begin{aligned} \mathcal{L} = & \bar{\psi}_1 \{ \gamma_0(\partial_\tau + ia_\tau) + \gamma_1 v_F(\partial_x + ia_x) + \gamma_2 v_F(\partial_y + ia_y) \} \psi_1 \\ & + (1 \leftrightarrow 2) + \frac{1}{2e^2} \left\{ \left(\frac{\partial a_x}{\partial y} - \frac{\partial a_y}{\partial x} \right)^2 + \left(\frac{\partial a_\tau}{\partial x} - \frac{\partial a_x}{\partial \tau} \right)^2 \right. \\ & \left. + \left(\frac{\partial a_\tau}{\partial y} - \frac{\partial a_y}{\partial \tau} \right)^2 \right\}. \end{aligned} \quad (25)$$

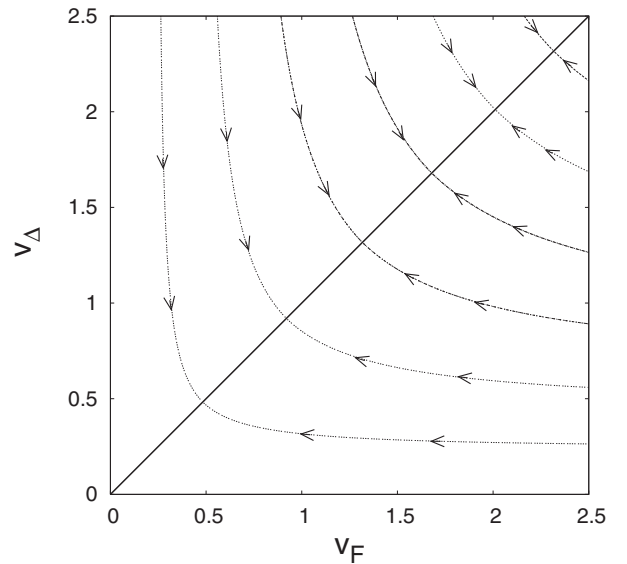


FIG. 8. The corresponding RG flow of v_F and v_Δ when charge renormalization is not taken into account. A line of fixed points at $v_F=v_\Delta$ is apparent.

By simple rescaling $\tau' = \tau$, $x = v_F x'$, $y = v_F y'$, $a'_\tau = a_\tau$, $a'_{x'} = v_F a_x$, and $a'_{y'} = v_F a_y$, we can transform this theory into a new theory in which the fermionic part of the action remains fully isotropic but an anisotropic Maxwellian term appears:

$$\frac{1}{2e^2 v_F^4} \left(\frac{\partial a'_{x'}}{\partial y'} - \frac{\partial a'_{y'}}{\partial x'} \right)^2 + \frac{1}{2e^2 v_F^2} \left\{ \left(\frac{\partial a'_{\tau'}}{\partial x'} - \frac{\partial a'_{x'}}{\partial \tau'} \right)^2 + \left(\frac{\partial a'_{\tau'}}{\partial y'} - \frac{\partial a'_{y'}}{\partial \tau'} \right)^2 \right\}. \quad (26)$$

Thus, the effect of having $v_F \neq c_s$ reduces to anisotropic couplings in the Maxwellian self-action of the gauge field. We denote these couplings as $e_\tau^2 \equiv e^2 v_F^4$ and $e_\perp^2 \equiv e^2 v_F^2$. These two couplings can be interpreted as two anisotropic charges. Such anisotropic charges can change the value of the critical number of fermions in the original theory, as already shown in Eq. (24). In contrast, within the isotropic QED₃, the critical number of fermions does not depend on e^2 . This is true as long as we have only one coupling constant, but once we introduce two different couplings this pleasing behavior is lost. This simple example shows that we can restore Lorentz invariance in the fermionic part of the action at the expense of breaking the Lorentz invariance of the Maxwellian part.

To summarize, the above discussion shows that at a bare level there are two intrinsic anisotropies in this problem; α is the fermionic one while $\theta = (e_\tau / e_\perp)$ is the Maxwellian one. e_τ and e_\perp are the couplings in the temporal and spatial directions, respectively. Thus at a bare level of the QED₃ theory relevant for the cuprates there are four coupling constants v_F , v_Δ , e_τ and e_\perp . We will now show that even though they contain anisotropy both in the fermionic and Maxwellian terms, in the large N limit the relativity is ultimately restored, without any assumptions about the size of the anisotropy.

To make good on the above claim notice that the effect of fermions on photons is still described by Eq. (8). However, the gauge-field stiffness is now

$$\Pi_{\mu\nu}^{(0)} = \frac{1}{2e^2} \bar{\epsilon}_{\chi\delta\mu} \bar{\epsilon}_{\chi\lambda\nu} k_\lambda k_\delta, \quad (27)$$

where the anisotropic Levi-Civita symbol is defined as $\bar{\epsilon}_{\tau\eta\mu} = \epsilon_{\tau\eta\mu}$, $\bar{\epsilon}_{x\eta\mu} = \theta \epsilon_{x\eta\mu}$, and $\bar{\epsilon}_{y\eta\mu} = \theta \epsilon_{y\eta\mu}$. Thus, the effective Lagrangian of the theory can be written as

$$\mathcal{L}_{\text{Aniso}} = \frac{1}{2} (\Pi_{\mu\nu}^{(0)} + \Pi_{\mu\nu}) a_\mu(k) a_\nu(-k), \quad (28)$$

where the effect of fermions has been introduced through the polarization function.

This expression allows us to find the renormalized couplings, comparing the original bare gauge-field stiffness with the screened one:

$$\Pi_{\mu\nu}^{(R)} = \Pi_{\mu\nu}^{(0)} + \Pi_{\mu\nu}. \quad (29)$$

We find that to the lowest order in $1/N$ the renormalized couplings are

$$\left(\frac{1}{e_\tau^2} \right)_R = \left(\frac{1}{e_\tau^2} \right) + \frac{N}{16k} v_F v_\Delta, \quad (30)$$

$$\left(\frac{1}{e_\perp^2} \right)_R = \left(\frac{1}{e_\perp^2} \right) + \frac{N}{16} \left(\frac{1}{k^{(1)} v_\Delta} + \frac{1}{k^{(2)} v_F} \right), \quad (31)$$

where $1/\bar{k} = 1/k^{(1)} + 1/k^{(2)}$ and $k^{(i)} = \sqrt{k_\alpha g_{\alpha\beta}^{(i)} k_\beta}$.

The above one-loop renormalization of the anisotropic charges allows us to set up the renormalization-group (RG) equations for the beta functions describing the flow of different couplings. These equations are rather complicated and we have been able to fully solve them only numerically. However, the following result is rather simple and can be extracted in an analytic form; on general grounds we expect that the nontrivial infrared fixed point should remain once we introduce the anisotropy, even though its position in parameter space may change. In order to find the value of the renormalized couplings at the fixed point, we analyze the *difference* between the renormalized couplings of the Maxwellian action

$$\left(\frac{1}{e_\tau^2} - \frac{1}{e_\perp^2} \right)_R = \left(\frac{1}{e_\tau^2} - \frac{1}{e_\perp^2} \right) + \frac{N}{16} \left[\frac{1}{k^{(1)}} \left(v_F v_\Delta - \frac{v_F}{v_\Delta} \right) + \frac{1}{k^{(2)}} \left(v_\Delta v_F - \frac{v_\Delta}{v_F} \right) \right]. \quad (32)$$

If we now rearrange our RG equations so as to focus on the beta function for this difference between the renormalized charges, $\beta_{e_\tau e_\perp}$, the above equation implies that

$$\beta_{e_\tau e_\perp} \sim \frac{N}{16} \left[\frac{\left(v_F v_\Delta - \frac{v_F}{v_\Delta} \right)}{k^{(1)}} + \frac{\left(v_\Delta v_F - \frac{v_\Delta}{v_F} \right)}{k^{(2)}} \right] + (\dots), \quad (33)$$

where (\dots) can be rewritten in terms of beta functions for all other couplings and thus must vanish at the putative fixed point. Clearly, noting that N is an arbitrarily large number, and that $k^{(i)}$ share the same sign, it follows that

$$v_F^{(R)} v_\Delta^{(R)} - \frac{v_F^{(R)}}{v_\Delta^{(R)}} = 0 \quad (34)$$

from where $v_\Delta^{(R)} = 1$ and given that the theory is fully invariant under the exchange $v_F \leftrightarrow v_\Delta$, it follows that $v_F^{(R)} = 1$. Putting this information back in the flow equations it is clear that $1/e_\tau^2$ and $1/e_\perp^2$ themselves diverge with the same slope at the fixed point and therefore their ratio $\theta = (e_\tau / e_\perp) \rightarrow 1$. This result shows that Lorentz invariance is restored and thus the previous results^{4,5,7} remain valid. However, we have made it clear that the physics behind the restoration of full Lorentz invariance follows a path more subtle than previously explored: at the infrared fixed point the relativity is restored due to the *interplay* between the velocity and charge renormalizations, the velocity renormalization by itself being insufficient to fully restore relativity of the theory.

III. CONCLUSIONS

We have proposed a simple criterion that allows an explicit computation of the critical number of fermions N_c in a theory that contains intrinsic anisotropies. We have checked,

at the Pisarski's level approximation, that this criterion captures the functional dependence of $N_c(v_\Delta) \sim 1/v_\Delta$ in the case in which an explicit expression can be obtained from analytic calculations.

Our criterion suggests that lattice simulations should be performed in such a way that important symmetries of the theory, namely, $v_F \leftrightarrow v_\Delta$, are protected. Otherwise, there is a danger of obtaining spurious results. In lattice QED₃, it seems worthwhile to investigate the existence of a possible confined phase in the region of the parameter space where fermionic velocities are small compared with the gauge-field velocity. Another venue that remains to be explored is the possible usefulness of similar criteria for the analysis of CSB or other nonperturbative phenomena in other physical systems that also feature anisotropic couplings.

Finally, we have shown that the velocity anisotropy in (2+1)QED does affect the number of critical fermion flavors N_c at which chiral symmetry is broken due to the phenomenon of mass generation, even though the large N theory remains fully relativistic in its critical phase. Surprisingly, N_c

is a nonmonotonic function of v_F and v_Δ , and, depending on the specific value of the ratio v_F/c_s , different regimes emerge. Our results imply that if phase fluctuations destroy the superconducting order in underdoped cuprates, we should expect a protected chirally symmetric critical phase—i.e., the pseudogap within this theory—as doping decreases, before we reach the antiferromagnetic region in the phase diagram. Details of how this sequence will take place depend on the specific value of the gauge-field velocity c_s for different compounds. We hope that our results will contribute to better understanding of the quantitative issues that surround the value of N_c in various effective theories and motivate further research on the anisotropic incarnations of the QED₃ theory.

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²⁵In Ref. 24 the authors claim that fermionic anisotropy goes to one, and thus they obtain the simplified model we used to show that this can be thought as a model in which there are anisotropic couplings in the Maxwellian term. Given that the number of fermions in physical systems is modest it would be interesting to analyze the consequences of having such anisotropic couplings.